

THE STAR OF DAVID RULE

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Abstract. In this note, a new concept called *SDR-matrix* is proposed, which is an infinite lower triangular matrix obeying the generalized rule of David star. Some basic properties of *SDR*-matrices are discussed and two conjectures on *SDR*-matrices are presented, one of which states that if a matrix is a *SDR*-matrix, then so is its matrix inverse (if exists).

Keywords: Narayana triangle, Pascal triangle, Lah triangle, *SDR*-matrix.

2000 MATHEMATICS SUBJECT CLASSIFICATION: Primary 05A10; Secondary 15A09

1. INTRODUCTION

The *Star of David rule* [13], originally stated by Gould in 1972, is given by

$$\binom{n}{k} \binom{n+1}{k-1} \binom{n+2}{k+1} = \binom{n}{k-1} \binom{n+1}{k+1} \binom{n+2}{k},$$

for any k and n , which implies that

$$\binom{n}{k+1} \binom{n+1}{k} \binom{n+2}{k+2} = \binom{n}{k} \binom{n+1}{k+2} \binom{n+2}{k+1}.$$

In 2003, the author observed in his Master dissertation [12] that if multiplying the above two identities and dividing by $n(n+1)(n+2)$, one can arrive at

$$N_{n,k+1} N_{n+1,k} N_{n+2,k+2} = N_{n,k} N_{n+1,k+2} N_{n+2,k+1},$$

where $N_{n,k} = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}$ is the Narayana number [9, A001263].

In the summer of 2006, the author asked Mansour [4] for a combinatorial proof of the above Narayana identity to be found. Later, by Chen's bijective algorithm for trees [1], Li and Mansour [3] provided a combinatorial proof of a general identity

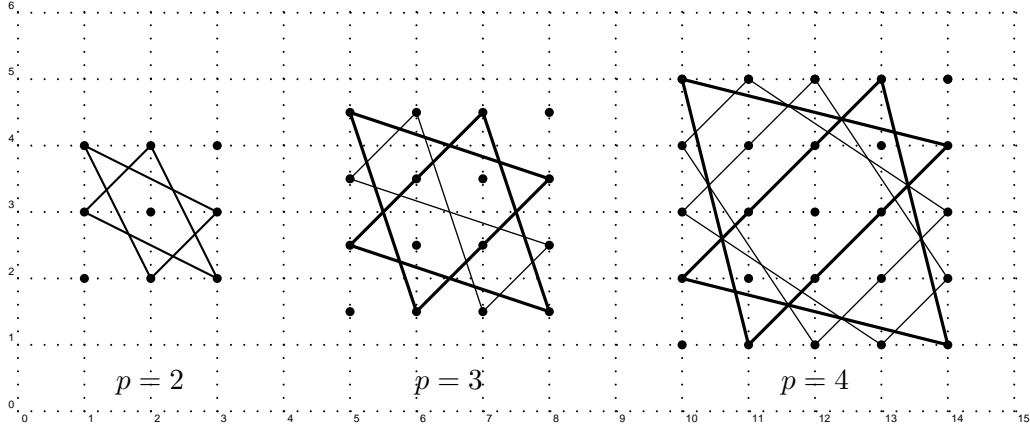
$$\begin{aligned} & N_{n,k+m-1} N_{n+1,k+m-2} N_{n+2,k+m-3} \cdots N_{n+m-2,k+1} N_{n+m-1,k} N_{n+m,k+m} \\ &= N_{n,k} N_{n+1,k+m} N_{n+2,k+m-1} \cdots N_{n+m-2,k+3} N_{n+m-1,k+2} N_{n+m,k+1}. \end{aligned}$$

This motivates the author to reconsider the Star of David rule and to propose a new concept called *SDR-matrix* which obeys the generalized rule of David star.

Definition 1.1. Let $\mathcal{A} = (A_{n,k})_{n \geq k \geq 0}$ be an infinite lower triangular matrix, for any given integer $m \geq 3$, if there hold

$$\prod_{i=0}^r A_{n+i,k+r-i} \prod_{i=0}^{p-r-1} A_{n+p-i,k+r+i+1} = \prod_{i=0}^r A_{n+p-i,k+p-r+i} \prod_{i=0}^{p-r-1} A_{n+i,k+p-r-i-1},$$

for all $2 \leq p \leq m-1$ and $0 \leq r \leq p-1$, then \mathcal{A} is called an *SDR-matrix* of order m .

FIGURE 1. The case $m = 5$.

In order to give a more intuitive view on the definition, we present a pictorial description of the generalized rule for the case $m = 5$. See Figure 1.

Let SDR_m denote the set of SDR -matrices of order m and SDR_∞ be the set of SDR -matrices \mathcal{A} of order ∞ , that is $\mathcal{A} \in SDR_m$ for any $m \geq 3$. By our notation, it is obvious that the Pascal triangle $\mathcal{P} = \left(\binom{n}{k} \right)_{n \geq k \geq 0}$ and the Narayana triangle $\mathcal{N} = \left(N_{n+1,k+1} \right)_{n \geq k \geq 0}$ are SDR -matrices of order 3. In fact, both of them will be proved to be SDR -matrices of order ∞ .

$$\mathcal{P} = \begin{pmatrix} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ 1 & 5 & 10 & 10 & 5 & 1 \\ \dots \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} 1 \\ 1 & 1 \\ 1 & 3 & 1 \\ 1 & 6 & 6 & 1 \\ 1 & 10 & 20 & 10 & 1 \\ 1 & 15 & 50 & 50 & 15 & 1 \\ \dots \end{pmatrix}.$$

In this paper, we will discuss some basic properties of the sets SDR_m and propose two conjectures on SDR_m for $3 \leq m \leq \infty$ in the next section. We also give some comments on relations between SDR -matrices and Riordan arrays in Section 3.

2. THE BASIC PROPERTIES OF SDR -MATRICES

For any infinite lower triangular matrices $\mathcal{A} = (A_{n,k})_{n \geq k \geq 0}$ and $\mathcal{B} = (B_{n,k})_{n \geq k \geq 0}$, define $\mathcal{A} \circ \mathcal{B} = (A_{n,k} B_{n,k})_{n \geq k \geq 0}$ to be the Hadamard product of \mathcal{A} and \mathcal{B} , denote by $\mathcal{A}^{\circ j}$ the j -th Hadamard power of \mathcal{A} ; If $A_{n,k} \neq 0$ for $n \geq k \geq 0$, then define $\mathcal{A}^{\circ(-1)} = (A_{n,k}^{-1})_{n \geq k \geq 0}$ to be the Hadamard inverse of \mathcal{A} .

From Definition 1.1, one can easily derive the following three lemmas.

Lemma 2.1. *For any $\mathcal{A} \in SDR_m$, $\mathcal{B} \in SDR_{m+i}$ with $i \geq 0$, there hold $\mathcal{A} \circ \mathcal{B} \in SDR_m$, and $\mathcal{A}^{\circ(-1)} \in SDR_m$ if it exists.*

Lemma 2.2. *For any $\mathcal{A} = (A_{n,k})_{n \geq k \geq 0} \in SDR_m$, then $(A_{n+i,k+j})_{n \geq k \geq 0} \in SDR_m$ for fixed $i, j \geq 0$.*

Lemma 2.3. *Given any sequence $(a_n)_{n \geq 0}$, let $A_{n,k} = a_n$, $B_{n,k} = a_k$ and $C_{n,k} = a_{n-k}$ for $n \geq k \geq 0$, then $(A_{n,k})_{n \geq k \geq 0}, (B_{n,k})_{n \geq k \geq 0}, (C_{n,k})_{n \geq k \geq 0} \in SDR_\infty$.*

Example 2.4. Let $a_n = n!$ for $n \geq 0$, then we have

$$\begin{aligned}\mathcal{P} &= (n!)_{n \geq k \geq 0} \circ (k!)_{n \geq k \geq 0}^{\circ(-1)} \circ ((n-k)!)_{n \geq k \geq 0}^{\circ(-1)}, \\ \mathcal{N} &= \left(\frac{1}{k+1}\right)_{n \geq k \geq 0} \circ \mathcal{P} \circ \left(\binom{n+1}{k}\right)_{n \geq k \geq 0}, \\ \mathcal{L} &= ((n+1)!)_{n \geq k \geq 0} \circ \mathcal{P} \circ ((k+1)!)_{n \geq k \geq 0}^{\circ(-1)},\end{aligned}$$

which, by Lemmas 2.1-2.3, produce that the Pascal triangle \mathcal{P} , the Narayana triangle \mathcal{N} and the Lah triangle \mathcal{L} belong to SDR_∞ , where $(\mathcal{L})_{n,k} = \binom{n}{k} \frac{(n+1)!}{(k+1)!}$ is the Lah number [2].

Theorem 2.5. *For any sequences $(a_n)_{n \geq 0}$, $(b_n)_{n \geq 0}$ and $(c_n)_{n \geq 0}$ such that $b_0 = 1$, $a_n \neq 0$ and $c_n \neq 0$ for $n \geq 0$, let $\mathcal{A} = (a_k b_{n-k} c_n)_{n \geq k \geq 0}$, then $\mathcal{A}^{-1} \in SDR_\infty$.*

Proof. By Lemmas 2.1 and 2.3, we have $\mathcal{A} \in SDR_\infty$. It is not difficult to derive the matrix inverse \mathcal{A}^{-1} of \mathcal{A} with the generic entries

$$(\mathcal{A}^{-1})_{n,k} = a_n^{-1} B_{n-k} c_k^{-1},$$

where B_n with $B_0 = 1$ are given by

$$(2.1) \quad B_n = \sum_{j=1}^n (-1)^j \sum_{i_1+i_2+\dots+i_j=n, i_1, \dots, i_j \geq 1} b_{i_1} b_{i_2} \dots b_{i_j}, \quad (n \geq 1).$$

Hence, by Lemmas 2.1 and 2.3, one can deduce that

$$\mathcal{A}^{-1} = (a_n^{-1})_{n \geq k \geq 0} \circ (B_{n-k})_{n \geq k \geq 0} \circ (c_k^{-1})_{n \geq k \geq 0} \in SDR_\infty,$$

as desired. \square

Specially, when $c_n := 1$ or $a_n := \frac{a_n}{n!}$, $b_n := \frac{b_n}{n!}$, $c_n := n!$, both $\mathcal{B} = (a_k b_{n-k})_{n \geq k \geq 0}$ and $\mathcal{C} = \left(\binom{n}{k} a_k b_{n-k}\right)_{n \geq k \geq 0}$ are in SDR_∞ , then so \mathcal{B}^{-1} and \mathcal{C}^{-1} . More precisely, let $a_n^{-1} = b_n^{-1} = c_n = n!(n+1)!$ for $n \geq 0$, note that the Narayana triangle $\mathcal{N} \in SDR_\infty$ and

$$N_{n+1,k+1} = \frac{1}{n+1} \binom{n+1}{k+1} \binom{n+1}{k} = \frac{n!(n+1)!}{k!(k+1)!(n-k)!(n-k+1)!}.$$

Then one has $\mathcal{N}^{-1} \in SDR_\infty$ by Theorem 2.5.

Theorem 2.5 suggests the following conjecture.

Conjecture 2.6. *For any $\mathcal{A} \in SDR_m$, if the inverse \mathcal{A}^{-1} of \mathcal{A} exists, then $\mathcal{A}^{-1} \in SDR_m$.*

Theorem 2.7. *For any sequences $(a_n)_{n \geq 0}$, $(b_n)_{n \geq 0}$ with $b_0 = 1$ and $a_n \neq 0$ for $n \geq 0$, let $\mathcal{A} = (a_n b_{n-k} a_k^{-1})_{n \geq k \geq 0}$, then the matrix power $\mathcal{A}^j \in SDR_\infty$ for any integer j .*

Proof. By Lemmas 2.1 and 2.3, we have $\mathcal{A} \in SDR_\infty$. Note that it is trivially true for $j = 1$ and $j = 0$ (where \mathcal{A}^0 is the identity matrix by convention). It is easy to obtain the

(n, k) -entries of \mathcal{A}^j for $j \geq 2$,

$$\begin{aligned} (\mathcal{A}^j)_{n,k} &= \sum_{k \leq k_{j-1} \leq \dots \leq k_1 \leq n} \mathcal{A}_{n,k_1} \mathcal{A}_{k_1,k_2} \cdots \mathcal{A}_{k_{j-2},k_{j-1}} \mathcal{A}_{k_{j-1},k} \\ &= a_n C_{n-k} a_k^{-1}, \end{aligned}$$

where C_n with $C_0 = 1$ is given by $C_n = \sum_{i_1+i_2+\dots+i_j=n, i_1, \dots, i_j \geq 0} b_{i_1} b_{i_2} \cdots b_{i_j}$ for $n \geq 1$.

By Lemmas 2.1 and 2.3, one can deduce that

$$\mathcal{A}^j = \left(a_n \right)_{n \geq k \geq 0} \circ \left(C_{n-k} \right)_{n \geq k \geq 0} \circ \left(a_k^{-1} \right)_{n \geq k \geq 0} \in SDR_\infty.$$

By Theorem 2.5 and its proof, we have $\mathcal{A}^{-1} \in SDR_\infty$ and $(\mathcal{A}^{-1})_{n,k} = a_n B_{n-k} a_k^{-1}$, where B_n is given by (2.1). Note that \mathcal{A}^{-1} has the form as required in Theorem 2.7, so by the former part of this proof, we have $\mathcal{A}^{-j} \in SDR_\infty$ for $j \geq 1$. Hence we are done. \square

Let $a_n = b_n = n!$, $a_n = b_n = n!(n+1)!$ or $a_n = n!(n+1)!$ and $b_n^{-1} = n!$ for $n \geq 0$ in Theorem 2.7, one has

Corollary 2.8. *For \mathcal{P} , \mathcal{N} and \mathcal{L} , then $\mathcal{P}^j, \mathcal{N}^j, \mathcal{L}^j \in SDR_\infty$ for any integer j .*

Remark 2.9. In general, for $\mathcal{A}, \mathcal{B} \in SDR_m$, their matrix product $\mathcal{A}\mathcal{B}$ is possibly not in SDR_m . For example, $\mathcal{P}, \mathcal{N} \in SDR_3$, but

$$\mathcal{P}\mathcal{N} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & 4 & 5 & 1 & & \\ & 8 & 18 & 9 & 1 & \\ & 16 & 56 & 50 & 14 & 1 \\ & 32 & 160 & 220 & 110 & 20 & 1 \\ & & & \dots & & & \end{pmatrix} \notin SDR_3.$$

Theorem 2.10. *For any $\mathcal{A} = (A_{n,k})_{n \geq k \geq 0}$ with $A_{n,k} \neq 0$ for $n \geq k \geq 0$, then $\mathcal{A} \in SDR_{m+1}$ if and only if $\mathcal{A} \in SDR_m$.*

Proof. Note that $SDR_{m+1} \subset SDR_m$, so the necessity is clear. It only needs to prove the sufficient condition. For the symmetry, it suffices to verify

$$\prod_{i=0}^r A_{n+i, k+r-i} \prod_{i=0}^{m-r} A_{n+m-i+1, k+r+i+1} = \prod_{i=0}^r A_{n+m-i+1, k+m-r+i+1} \prod_{i=0}^{m-r} A_{n+i, k+m-r-i},$$

for $0 \leq r \leq [m/2]-1$. We just take the case $r = 0$ for example, others can be done similarly. It is trivial when $A_{n, k+m} = A_{n+1, k+m+1} = 0$. So we assume that $A_{n, k+m} \neq 0, A_{n+1, k+m+1} \neq 0$, then all $A_{n+i, k+j}$ to be considered, except for $A_{n, k+m+1}$, must not be zero. By Definition 1.1, we have

$$\begin{aligned} & A_{n+m-i, k+i} A_{n+m-i-1, k+i+1} A_{n+m-i+1, k+i+2} \\ (2.2) \quad &= A_{n+m-i+1, k+i+1} A_{n+m-i, k+i+2} A_{n+m-i-1, k+i}, \quad (0 \leq i \leq m-1). \end{aligned}$$

$$(2.3) \quad A_{n+m+1, k+m+1} \prod_{i=0}^{m-1} A_{n+i, k+m-i} = A_{n+1, k+1} \prod_{i=0}^{m-1} A_{n+m-i+1, k+i+2}.$$

$$(2.4) \quad A_{n+1, k+1} \prod_{i=0}^{m-1} A_{n+m-i, k+i+1} = A_{n+m, k+m} \prod_{i=0}^{m-1} A_{n+i+1, k+m-i-1}.$$

$$(2.5) \quad A_{n+m,k+m} \prod_{i=0}^{m-1} A_{n+i,k+m-i-1} = A_{n,k} \prod_{i=0}^{m-1} A_{n+m-i,k+i+1}.$$

Multiplying (2.2)–(2.5) together, after cancellation, one can get

$$A_{n,k} \prod_{i=0}^m A_{n+m-i+1,k+i+1} = A_{n+m+1,k+m+1} \prod_{i=0}^m A_{n+i,k+m-i},$$

which confirms the case $r = 0$. \square

Remark 2.11. The condition $A_{n,k} \neq 0$ for $n \geq k \geq 0$ in Theorem 2.10 is necessary. The following example verifies this claim.

$$\left(\binom{\frac{n+k}{2}}{\frac{n-k}{2}} \right)_{n \geq k \geq 0} = \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ 0 & 2 & 0 & 1 & & \\ 1 & 0 & 3 & 0 & 1 & \\ 0 & 3 & 0 & 4 & 0 & 1 \\ & & \dots & & & \end{pmatrix} \in SDR_3, \text{ but not in } SDR_4.$$

Recall that the Narayana number $\mathcal{N}_{n+1,k+1}$ can be represented as

$$\mathcal{N}_{n+1,k+1} = \frac{1}{n+1} \binom{n+1}{k+1} \binom{n+1}{k} = \det \begin{pmatrix} \binom{n}{k} & \binom{n}{k+1} \\ \binom{n+1}{k} & \binom{n+1}{k+1} \end{pmatrix},$$

so we can come up with the following definition.

Definition 2.12. Let $\mathcal{A} = (A_{n,k})_{n \geq k \geq 0}$ be an infinite lower triangular matrix, for any integer $j \geq 1$, define $\mathcal{A}_{[j]} = (A_{n,k}^{[j]})_{n \geq k \geq 0}$, where

$$A_{n,k}^{[j]} = \det \begin{pmatrix} A_{n,k} & \cdots & A_{n,k+j-1} \\ \vdots & \dots & \vdots \\ A_{n+j-1,k} & \cdots & A_{n+j-1,k+j-1} \end{pmatrix}.$$

Theorem 2.13. For any sequences $(a_n)_{n \geq 0}$, $(b_n)_{n \geq 0}$ and $(c_n)_{n \geq 0}$ such that $b_0 = 1$, $a_n \neq 0$ and $c_n \neq 0$ for $n \geq 0$, let $\mathcal{A} = (a_k b_{n-k} c_n)_{n \geq k \geq 0}$, then $\mathcal{A}_{[j]} \in SDR_\infty$ for any integer $j \geq 1$.

Proof. By Lemmas 2.1 and 2.3, we have $\mathcal{A} \in SDR_\infty$. It is easy to derive the determinant

$$\det \begin{pmatrix} a_k b_{n-k} c_n & \cdots & a_{k+j-1} b_{n-k-j+1} c_n \\ \vdots & \dots & \vdots \\ a_k b_{n-k+j-1} c_{n+j-1} & \cdots & a_{k+j-1} b_{n-k} c_{n+j-1} \end{pmatrix} = B_{n-k} \prod_{i=0}^{j-1} a_{k+i} c_{n+i},$$

where B_n with $B_0 = 1$ are given by

$$B_n = \det \begin{pmatrix} b_n & \cdots & b_{n-j+1} \\ \vdots & \dots & \vdots \\ b_{n+j-1} & \cdots & b_n \end{pmatrix}.$$

Hence, by Lemmas 2.1 and 2.3, one can deduce that

$$\mathcal{A}_{[j]} = \left(\prod_{i=0}^{j-1} a_{k+i} \right)_{n \geq k \geq 0} \circ (B_{n-k})_{n \geq k \geq 0} \circ \left(\prod_{i=0}^{j-1} c_{n+i} \right)_{n \geq k \geq 0} \in SDR_{\infty},$$

as desired. \square

Let $a_n^{-1} = b_n^{-1} = c_n = n!$, $a_n^{-1} = b_n^{-1} = c_n = n!(n+1)!$ or $a_n^{-1} = c_n = n!(n+1)!$ and $b_n^{-1} = n!$ for $n \geq 0$ in Theorem 2.13, one has

Corollary 2.14. *For \mathcal{P} , \mathcal{N} and \mathcal{L} , then $\mathcal{P}_{[j]}, \mathcal{N}_{[j]}, \mathcal{L}_{[j]} \in SDR_{\infty}$ for any integer $j \geq 1$.*

Theorem 2.13 suggests the following conjecture.

Conjecture 2.15. *If $\mathcal{A} \in SDR_{\infty}$, then $\mathcal{A}_{[j]} \in SDR_{\infty}$ for any integer $j \geq 1$.*

Remark 2.16. The conjecture on SDR_m is generally not true for $3 \leq m < \infty$. For example, let $\mathcal{A} = (A_{n,k})_{n \geq k \geq 0}$ with $A_{n,k} = \left(\frac{n+k}{\frac{n-k}{2}} \right)$, then we have $\mathcal{A} \in SDR_3$, but

$$\mathcal{A}_{[2]} = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ \mathbf{2} & -2 & 1 & & \\ -2 & 6 & -\mathbf{3} & 1 & \\ 3 & -\mathbf{9} & 12 & -4 & 1 \\ & & \dots & & \end{pmatrix} \notin SDR_3, \quad \mathcal{A}_{[3]} = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 2 & 0 & 1 & & \\ 0 & 15 & 0 & 1 & \\ 9 & 0 & 36 & 0 & 1 \\ & & \dots & & \end{pmatrix} \in SDR_3.$$

3. FURTHER COMMENTS

We will present some further comments on the connections between SDR -matrices and Riordan arrays. The concept of Riordan array introduced by Shapiro et al [8], plays a particularly important role in studying combinatorial identities or sums and also is a powerful tool in study of many counting problems [5, 6, 7]. For examples, Sprugnoli [6, 10, 11] investigated Riordan arrays related to binomial coefficients, colored walks, Stirling numbers and Abel-Gould identities.

To define a Riordan array we need two analytic functions, $d(t) = d_0 + d_1t + d_2t^2 + \dots$ and $h(t) = h_1t + h_2t^2 + \dots$. A *Riordan array* is an infinite lower triangular array $\{d_{n,k}\}_{n,k \in \mathbb{N}}$, defined by a pair of formal power series $(d(t), h(t))$, with the generic element $d_{n,k}$ satisfying

$$d_{n,k} = [t^n]d(t)(h(t))^k, \quad (n, k \geq 0).$$

Assume that $d_0 \neq 0 \neq h_1$, then $(d(t), h(t))$ is an element of the *Riordan group* [8], under the group multiplication rule:

$$(d(t), h(t))(g(t), f(t)) = (d(t)g(h(t)), f(h(t))).$$

This indicates that the identity is $I = (1, t)$, the usual matrix identity, and that

$$(d(t), h(t))^{-1} = \left(\frac{1}{d(\bar{h}(t))}, \bar{h}(t) \right),$$

where $\bar{h}(t)$ is the compositional inverse of $h(t)$, i.e., $\bar{h}(h(t)) = h(\bar{h}(t)) = t$.

By our notation, we have

$$\begin{aligned}
\mathcal{P} &= \left(\frac{1}{1-t}, \frac{t}{1-t} \right) \in SDR_{\infty}, \\
\mathcal{P}^j &= \left(\frac{1}{1-jt}, \frac{t}{1-jt} \right) \in SDR_{\infty}, \\
\left(\left(\frac{\frac{n+k}{2}}{\frac{n-k}{2}} \right) \right)_{n \geq k \geq 0} &= \left(\frac{1}{1-t^2}, \frac{t}{1-t^2} \right) \in SDR_3, \\
\left(\frac{1}{1-t^2}, \frac{t}{1-t^2} \right)^{-1} &= \left(\frac{1-\sqrt{1-4t^2}}{2t^2}, \frac{1-\sqrt{1-4t^2}}{2t} \right) \in SDR_3, \\
\left(d_{n-k} \right)_{n \geq k \geq 0} &= (d(t), t) \in SDR_{\infty}.
\end{aligned}$$

Hence, it is natural to ask the following question.

Question 3.1. *Given a formal power series $d(t)$, what conditions $h(t)$ should satisfy, such that $(d(t), h(t))$ forms an SDR-matrix.*

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